INSTITUTE OF PHYSICS PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **23** (2002) 565–568 PII: S0143-0807(02)36749-7

Note on the relativistic elastic head-on collision

Hanno Essén

Department of Mechanics, KTH, SE-100 44 Stockholm, Sweden

E-mail: hanno@mech.kth.se

Received 9 May 2002, in final form 24 June 2002
Published 6 September 2002
Online at stacks.iop.org/EJP/23/565

Abstract
A short, elegant, instructive, and explicit solution for the relativistic elastic head-on collision is presented. It uses the invariance of the relative speed and a suitable Lorentz transformation of the total momentum four-vector before and after the collision.

An important application of special relativity is in the study of high-energy collisions. For pedagogical reasons it is instructive to compare the outcome of relativistic calculations with non-relativistic Newtonian results. Unfortunately, the relativistic problems often involve quite tough algebra. Therefore even the simplest of all collision problems, that of the elastic head-on collision, is rarely solved explicitly in textbooks. Among the many excellent textbooks that I have consulted [1–8] only Smith [6] presents an explicit solution, but it is fairly lengthy. The brief and elegant solution presented below may therefore be of pedagogical interest.

First some background theory and notation [4]. We denote a particle four-trajectory as

$$\boldsymbol{R}(\tau) = [\boldsymbol{r}(\tau), ct(\tau)] = [x(\tau), y(\tau), z(\tau), ct(\tau)] \tag{1}$$

where τ is the proper time:

$$\mathrm{d}\tau = \mathrm{d}t/\gamma(u), \tag{2}$$

$\boldsymbol{u} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ is the three-velocity, and

$$\gamma(u) \equiv \frac{1}{\sqrt{1 - u^2/c^2}} \tag{3}$$

is the gamma factor. The four-velocity is then defined:

$$\boldsymbol{U} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}\tau} = \gamma(u)[\boldsymbol{u}, c]. \tag{4}$$

By definition, the Minkowski length squared of the four-velocity,

$$\boldsymbol{U}^2 = -U_x^2 - U_y^2 - U_z^2 + U_t^2 = \gamma^2(u)(-\boldsymbol{u}^2 + c^2) = c^2, \tag{5}$$

is always c^2.

If the rest mass of the particle is m, its four-momentum is defined by

$$\boldsymbol{P} = m\boldsymbol{U} = \gamma(u)[\boldsymbol{p}, mc], \tag{6}$$

where $\boldsymbol{p} = m\boldsymbol{u}$ is the three-momentum. The length squared of the four-momentum is

$$\boldsymbol{P}^2 = m^2c^2, \tag{7}$$

in view of equation (5), and is thus a constant scalar.

The components of all these four-vectors, $\boldsymbol{R}$, $\boldsymbol{U}$, and $\boldsymbol{P}$, transform under Lorentz transformations in the same way. For a standard Lorentz transformation from a system S to another system S' moving with speed v along the positive x-axis of S, their origins coinciding at $t = 0$, we have, for the components of $\boldsymbol{R}'$,

$$x' = \gamma(v)\left(x - \frac{v}{c}ct\right), \qquad y' = y, \qquad z' = z, \qquad ct' = \gamma(v)\left(ct - \frac{v}{c}x\right), \tag{8}$$

in terms of those of $\boldsymbol{R}$. Corresponding formulae are valid for the components of the other four-vectors. For example, we have

$$P'_x = \gamma(v)\left(P_x - \frac{v}{c}P_t\right), \qquad P'_y = P_y, \qquad P'_z = P_z, \qquad P'_t = \gamma(v)\left(P_t - \frac{v}{c}P_x\right), \tag{9}$$

for the components of the four-momentum.

Collision problems are best approached using the constancy of the total four-momentum:

$$\overline{\boldsymbol{P}} = \sum \boldsymbol{P}_i = \text{ constant.} \tag{10}$$

The constancy of the three-vector part is the generalization of the conservation of Newtonian three-momentum, while the constancy of the time (or fourth) component is the relativistic generalization of the conservation of energy. Assume then that we have two particles, with four-momenta $\boldsymbol{P} = M\gamma(u)[\boldsymbol{u}, c]$ and $\boldsymbol{Q} = m\gamma(w)[\boldsymbol{w}, c]$, before the collision. After an elastic collision of these, by definition, the number of particles and their rest masses are unchanged. The constancy of $\overline{\boldsymbol{P}}$ then gives

$$\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{P}' + \boldsymbol{Q}' \tag{11}$$

or, equivalently,

$$M\gamma(u)[\boldsymbol{u}, c] + m\gamma(w)[\boldsymbol{w}, c] = M\gamma(u')[\boldsymbol{u}', c] + m\gamma(w')[\boldsymbol{w}', c], \tag{12}$$

where the primed quantities are those after the collision. Now consider the square of both sides. Since $\boldsymbol{P}^2 = \boldsymbol{P}'^2 = M^2c^2$, and similarly for $\boldsymbol{Q}$, we find that

$$\boldsymbol{P} \cdot \boldsymbol{Q} = \boldsymbol{P}' \cdot \boldsymbol{Q}'. \tag{13}$$

To calculate this invariant ($\boldsymbol{P} \cdot \boldsymbol{Q} = -P_x Q_x - P_y Q_y - P_z Q_z + P_t Q_t$) we can use any reference frame we like. If M is the bullet particle and m the target, it is natural to use the rest frame of the target (the laboratory frame), where $\boldsymbol{w} = 0$. We then get

$$\boldsymbol{P} \cdot \boldsymbol{Q} = \boldsymbol{P}' \cdot \boldsymbol{Q}' = Mm\gamma(u)c^2. \tag{14}$$

Since this quantity has the same value after the collision, we realize that the speed u here must have an invariant meaning. It is the *relative speed*, defined as the speed of one of the particles relative to the rest frame of the other particle. This is the most basic result in a two-particle elastic collision: the relative speed is unchanged. Note the use of the word *speed*; the relative velocity vector can change direction, but its length, the speed, is conserved.

We now specialize to a head-on collision; that is, we assume that all velocities are along a line (the x-axis). We first calculate the total four-momentum of the system in the rest frame, S, of m and get

$$\overline{\boldsymbol{P}} = \boldsymbol{P} + \boldsymbol{Q} = [M\gamma(u)u, 0, 0, (M\gamma(u) + m)c], \tag{15}$$

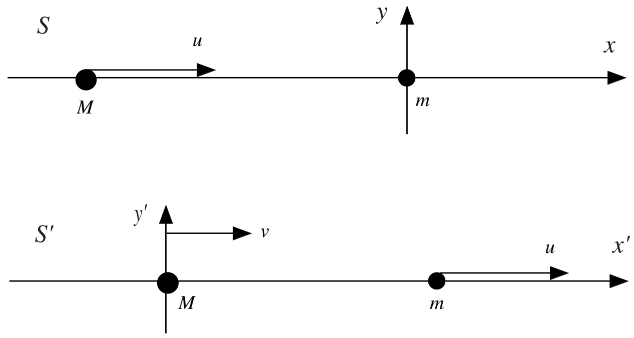

Figure 1. The situation before and after collision. Before the collision the system is seen in the rest frame S of the target m, and after the collision in the rest frame S' of the bullet M. The speed v of S' is the speed of M after the collision.

where M has speed u and moves in the direction of the positive x-axis; see figure 1. After the collision the relative speed is still u and the main trick here is to find the total four-momentum after the collision in the frame S' in which M is at rest and thus m moves with speed u. In S' the total four-momentum is

$$\overline{\boldsymbol{P'}} = [m\gamma(u)u, 0, 0, (M + m\gamma(u))c]; \tag{16}$$

see figure 1. The speed v of S' relative to S must be the speed of M after the collision. The total four-momenta $\overline{\boldsymbol{P}}$ and $\overline{\boldsymbol{P'}}$ are not equal, since they refer to different frames, but they must be related by a Lorentz transformation (9). This gives us

$$m\gamma(u)u = \gamma(v)\left(M\gamma(u)u - \frac{v}{c}(M\gamma(u) + m)c\right), \tag{17}$$

for the P'_x-component, and

$$(M + m\gamma(u))c = \gamma(v)\left((M\gamma(u) + m)c - \frac{v}{c}M\gamma(u)u\right), \tag{18}$$

for the P'_t-component.

We now wish to find the velocity v of M after the collision. If we combine the two equations (17) and (18) we can eliminate $\gamma(v)$ and get the linear equation

$$m\gamma u\left((M\gamma + m)c - \frac{v}{c}M\gamma u\right) = (M + m\gamma)c(M\gamma u - v(M\gamma + m)), \tag{19}$$

where $\gamma \equiv \gamma(u)$, for v. If we solve for v we get

$$v(u) = \frac{(M^2 - m^2)u}{M^2 + m^2 + 2Mm/\gamma(u)}, \tag{20}$$

after some simplification by hand. The non-relativistic result is obtained by putting $\gamma(u) = 1$ and is

$$v_0(u) = \frac{M - m}{M + m}u. \tag{21}$$

If we let $u \to c$ in (20) we get

$$v(c) = \frac{M^2 - m^2}{M^2 + m^2}c, \tag{22}$$

for the ultrarelativistic limit.

Equation (20) shows that the bullet will be at rest after the collision if it has the same mass as the target ($M = m$), and that it will continue to move in the positive x-direction after the collision if it has greater mass than the target ($M > m$) but will bounce back towards the negative x-direction in the opposite case ($M < m$). These results agree with non-relativistic intuition. Finally, one can show that for $u \to c$ and similar masses ($M \approx m$), the relativistic v is twice as large as the non-relativistic v_0. Since the neutron mass $m_{\rm n} = 1.008\,6650$ u is slightly larger than the proton mass $m_{\rm p} = 1.007\,2765$ u we find that ultrarelativistic protons ($u \approx c$) that collide elastically and head-on with neutrons recoil with $v(c) \approx 0.0014\,c$. The non-relativistic result is $v_0(c) \approx 0.0007\,c$. The elastic head-on collisions are of course only a small fraction of the real collisions, so this effect is probably not easy to observe experimentally.

This concludes our treatment of the relativistic elastic head-on collision. The main trick that solved the problem was finding the total four-momentum in two different reference frames, before and after the collision: first the rest frame S of the target and then the rest frame S' of the bullet. Use of the conservation of relative speed and a Lorentz transformation then reduces the problem to solving a linear equation.

References

[1] Landau L and Lifshitz E M 1975 *The Classical Theory of Fields* 4th edn (Oxford: Pergamon)
[2] Laurent B 1994 *An Introduction to Spacetime, a First Course on Relativity* (Singapore: World Scientific)
[3] Rindler W 1991 *Introduction to Special Relativity* 2nd edn (Oxford: Oxford University Press)
[4] Rindler W 2001 *Relativity, Special, General, and Cosmological* (Oxford: Oxford University Press)
[5] Shadowitz A 1988 *Special Relativity* (New York: Dover)
[6] Smith J H 1995 *Introduction to Special Relativity* (New York: Dover)
[7] Stephenson G and Kilminster C W 1987 *Special Relativity for Physicists* (New York: Dover)
[8] Taylor E F and Wheeler J A 1992 *Spacetime Physics* 2nd edn (San Francisco, CA: Freeman)