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LETTERS AND COMMENTS

The game of the 'very small' and the 'very big'

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Abstract
Measurement of the length of a moving rod by a single observer is reconsidered, showing that the results depend on the distance between rod and observer and on the angle under which one of the ends is seen.

In the case of a very *big distance* and a *very small* distance the conditions under which the Lorentz contraction can be detected is discussed, showing that results obtained in Manoukian and Sukkasena (Manoukian E B and Sukkasena S 2002 *Eur. J. Phys.* **23** 103) are the consequences of simplifying assumptions.

When speaking about a given physical quantity, it is good to know who measures it, when and where the measurement is performed and the device used. During the derivation of an equation which accounts for the result of a measurement, simplifying assumptions are made which sometimes are not always mentioned when physicists use them. The purpose of our paper is to show that some of the formulae presented by Manoukian and Sukkasena [1] are the result of simplifying assumptions.

Weinstein [2] shows how a single observer can measure the length of a moving rod. The problem is solved only for the case when the distance between the observer and the moving rod is 'very small'. In particular, the problem is intimately connected with the 'visibility' or the 'invisibility' of the Lorentz contraction, tackled by Terrel [3] and recently discussed in this journal [1].

Consider a rod of proper length L_0 at rest in the $S'(x'O'y')$ reference frame (figure 1). It is parallel to the $O'x'$ axis and positioned apart from it at a distance y'. An observer O' located at the origin O' measures its length. Let $2'(x'_2, y') = 2'(r'_2\cos\theta'_2, r'_2\sin\theta'_2)$ be the leading edge of the rod and $1'(x'_1, y') = 1'(r'_1\cos\theta'_1, r'_1\sin\theta'_1)$ its trailing edge. Observer O' sees the light from both ends of the rod at a single time $t' = 0$. Light originating from edge 2 has left it at $-r'_2/c$ whereas light originating from $1'$ has left it at a moment $-r'_1/c$. The events involved in the length measurement by O' are $1'(r'_1\cos\theta'_1, r'_1\sin\theta'_1, -r'_1/c)$ and $2'(r'_2\cos\theta'_2, r'_2\sin\theta'_2, -r'_2/c)$. Consider now a second reference frame $S(xOy)$ relative to which the reference frame $S(x'O'y')$ and the rod move with constant velocity $v = \beta c$ in the positive direction of the common

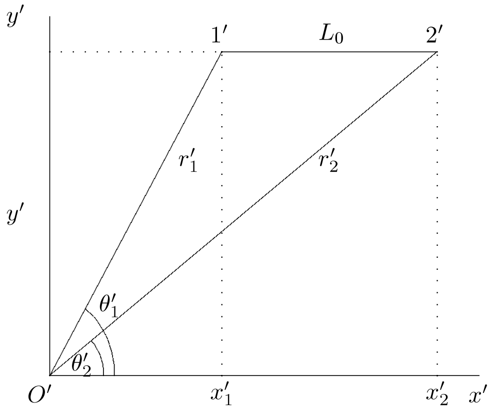

Figure 1. Location of the rod in its rest frame $S'(x'O'y')$.

$Ox(O'x')$ axes. The axes of the two reference frames are parallel to each other. At a moment $t = t' = 0$ the origins of S and S' are located at the same point in space. An observer O located at the origin O of the S frame receives light signals which have left the edges $1'$ and $2'$ at an instant of time $t = 0$. From the observer's point of view events $1'$ and $2'$ are characterized by the spacetime coordinates $1(x_1, y, -r_1/c) = 1(r_1\cos\theta_1, r_1\sin\theta_1, -r_1/c)$ and $2(x_2, y, -r_2/c) = 2(r_2\cos\theta_2, r_2\sin\theta_2, -r_2/c)$. In accordance with the Lorentz–Einstein transformations we should have

$$x_1 = \gamma(x_1' - \beta r_1') \tag{1}$$

and

$$x_2 = \gamma(x_2' - \beta r_2'). \tag{2}$$

By definition $L_0 = x_2' - x_1'$ and $L = x_2 - x_1$, with L representing the length of the moving rod as measured by observer O following the measurement procedure. From equations (2) and (3) we obtain that L and L_0 are related by

$$L = \gamma L_0 - \beta\gamma(r_2' - r_1'). \tag{3}$$

Application of the cosine theorem gives (figure 1)

$$r_2' = \sqrt{r_1'^2 + L_0^2 + 2r_1' L_0\cos\theta_1'} = r_1'\sqrt{1 + \left(\frac{L_0}{r_1'}\right)^2 + 2\left(\frac{L_0}{r_1'}\right)\frac{\cos\theta_1 + \beta}{1 + \beta\cos\theta_1}}. \tag{4}$$

Equation (4) becomes

$$L = \gamma\left\{L_0 - \beta r_1'\left[\sqrt{1 + \left(\frac{L_0}{r_1'}\right)^2 + 2\left(\frac{L_0}{r_1'}\right)\frac{\cos\theta_1 + \beta}{1 + \beta\cos\theta_1}} - 1\right]\right\} \tag{5}$$

where we have taken into account that the angles θ and θ' are related by the aberration of light effect formula:

$$\cos\theta' = \frac{\cos\theta_1 + \beta}{1 + \beta\cos\theta_1}. \tag{6}$$

Equation (5) is an exact formula free of simplifying assumptions. Weinstein [2] considers only the case of a rod standing very close to the observer. In that case $\theta_1 = 0$ (both edges of the rod are 'outgoing') or $\theta_1 = \pi$ (both edges of the rod are 'incoming'). Equation (5) leads in the first case to

$$L_{\theta_1=0} = \gamma L_0(1 - \beta) \tag{7}$$

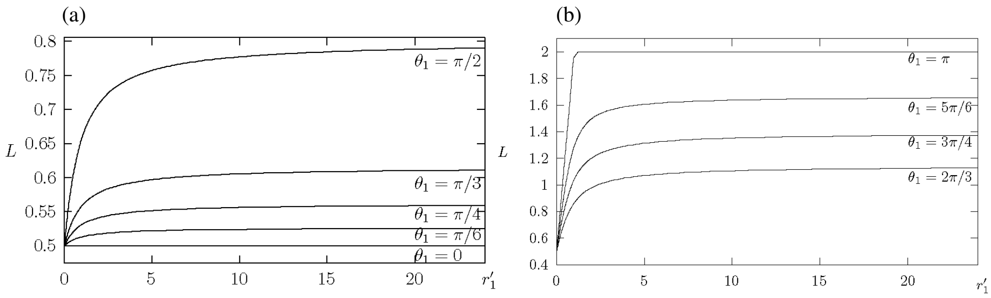

Figure 2. (a) The measured length of the moving rod as a function of the proper length r'_1 of its trailing end 1 for different values of the angle θ_1 under which it is seen from the $S(xOy)$ reference frame, when end 1 is outgoing. (b) The measured length of the moving rod as a function of the proper length r'_1 of its trailing end 1 for different values of the angle θ_1 under which it is seen from the $S(xOy)$ reference frame, when end 1 is incoming.

whereas in the second case it leads to

$$L_{\theta_1=\pi} = \gamma L_0(1+\beta). \tag{8}$$

We say that in this case 'the very small distance approximation' is made. Astronomers would rather consider the case when the distance between observer O' and edge $1'$ is very large, ($r'_1 \to \infty$). In that case equation (5) leads to

$$L_(r_1 \to \infty) = L_0 \frac{1-\beta\cos\theta'_1}{\sqrt{1-\beta^2}} = L_0 \frac{\sqrt{1-\beta^2}}{1+\beta\cos\theta_1} \tag{9}$$

often encountered in the literature of the subject without its approximate character being mentioned. In figure 2 we present the variation of L with r'_1 for a constant value of the angle θ_1 under which the edge 1 is seen by observer O considering that the rod moves relative to him with velocity $\beta = 0.6$ and for $L_0 = 1$.

In figure 2(a) we have chosen such values of the angle θ_1 for which edge 1 is 'outgoing' whereas in figure 2(b) values for which edge 1 is incoming are chosen. Figure 2 illustrates the transition from the 'very small distance' to the 'very large distance' approximations. We can now answer the much discussed question of the 'visibility' of the Lorentz contraction. In the case of the 'very small distance' approximation (equations (8) and (9)) we consider that $L_{0,i} = \alpha L_0 (\alpha < 1)$ is the proper length of the incoming part whereas $L_{0,0} = (1-\alpha)L_0$ is the length of the outgoing part. The visibility condition requires that

$$L_0\gamma^{-1} = \gamma L_0\alpha(1-\beta) + \gamma(1-\alpha)L_0(1+\beta) \tag{10}$$

from where we obtain for α the value

$$\alpha = \frac{1+\beta}{2} < 1. \tag{11}$$

In the case of the 'very large distance' approximation we impose the condition

$$L_0\gamma^{-1} = \gamma L_0(1-\beta\cos\theta'_1) = \gamma^{-1}L_0(1+\beta\cos\theta_1)^{-1} \tag{12}$$

from where we obtain

$$\cos\theta'_1 = \beta \tag{13}$$

$$\cos\theta_1 = 0 \tag{14}$$

in good agreement with the aberration of light effect and with the results obtained by Manoukian and Sukkasena [1], showing that the derivation they present is not free of simplifying assumptions.

References

[1] Manoukian E B and Sukkasena S 2002 *Eur. J. Phys.* **23** 103
[2] Weinstein R 1960 *Am. J. Phys.* **28** 607
[3] Terrel J 1959 *Phys. Rev.* **4** 1041